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SECURE DOMINATION IN TRANSFORMATION GRAPH G^{xy+}

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ABSTRACT. In this paper, we characterize graphs for which the secure domination number of the transformation graph G^{xy+} is 1 or 2. Also we prove that for any connected graph G with at least 4 pendant vertices, the secure domination number is greater than or equal to the secure domination number of the transformation graph G^{-++}. We also find a bound for the secure domination number of G^{--+} when G is a tree.

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1. INTRODUCTION

Let $G = (V(G), E(G))$ be a simple non-trivial connected graph with n vertices and m edges. The *open neighborhood* of $v \in V(G)$ is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. An *S-external private neighbour* of $v \in S$ is a vertex $u \in V(G) \setminus S$ such that u is adjacent to only v from the set S. The set of all S-external private neighbours of v is called *S-external private neighbour set* denoted by $epn(v, S)$. A graph G is a *complete bipartite graph* if its vertex set can be divided into two disjoint sets, V_1 and V_2, such that every vertex of V_1 is adjacent to every vertex of V_2, and no two vertices within each partite set is adjacent to each other. The line graph L(G) of G has its vertex set as $E(G)$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent in G.

A set $D \subseteq V(G)$ is a dominating set of G if each vertex in $V(G)\setminus D$ is adjacent to at least one vertex in D. The minimum cardinality of a dominating set in G is the domination number of G denoted by $\gamma(G)$ [5].

Domination in graph theory can be used to solve a wide range of mathematical and practical issues, such as monitoring communication and electricity networks, locating infrastructure, and defending a territory or an area. A graph can be used to solve such problems if each vertex v represents some location and the adjacency implies that there is direct access between the vertices. In order to defend a particular area, one or more guards can be strategically placed at each vertex v, and a guard at each vertex can protect every vertex in its neighbourhood. As a result, concepts like Roman domination, secure domination, and co-secure domination were developed. According to the concept of *secure domination*, a guard should be positioned at each vertex of $S \subseteq V(G)$ such that S is a dominating set of G and for each $u \in V(G)\setminus D$ there exists $v \in S$ such that $(S\setminus\{v\}) \cup \{u\}$ is a dominating set in G [3]. This concept was first explained by Cockayne, Grobler,

Grundlingh, Munganga and Van Vuuren [3] and is extensively studied in case of join of graphs by Castillano and Ugbinada [7]. Behzad [2] studied the criterion for the planarity of a transformation graph called the total graph and also found a characterization in terms of planarity. Wu and Meng [8] generalized the concept of total graph and introduced new eight graphical transformations. Later Wu and Zhang [9] studied one out of the eight transformation graph G^{xyz} when xyz = -++.
The concept of transformation graphs was extensively studied for many decades, however, when we try to extend its usability to a larger area of coverage, a major question arises -

Having known the given graph, can one determine all the properties of the transformed graph ?

In answer to this question, Jebitha and Joseph [6] obtained results for transformation G^{+-+} in terms of domination number.

The *transformation graph* G^{xyz} of G is defined on the vertex set $V(G) \cup E(G)$. Let a, b be two vertices in $V(G^{xyz})$ and the associativity of a and b is + if they are adjacent or incident in G and - otherwise. The vertices a and b correspond to the first term x (resp. the second term y or the third term z) of xyz if both a and b are in $V(G)$ (resp. both a and b are in $E(G)$, or one of a and b is in $V(G)$ and the other is in $E(G)$). Two vertices a and b of G^{xyz} are joined by an edge if and only if their associativity in G is consistent with the corresponding term of xyz [8]. Notice that there are eight different

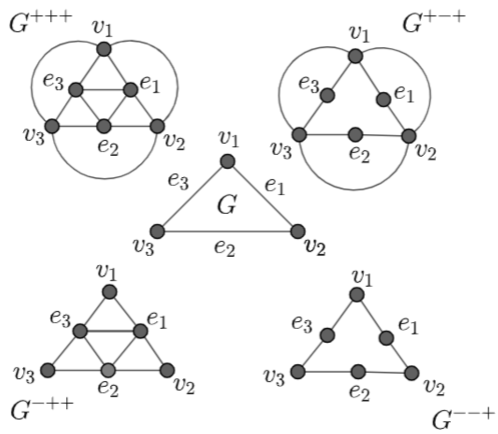

Figure 1. A graph G and its transformation graph G^{xy+}

transformations namely G^{+++}, G^{+--}, G^{++-}, G^{+-+}, G^{-++}, G^{--+}, G^{-+-} and G^{---}. A graph G and its 4 different transformation graphs are given in Figure 1.

In this paper we study the secure domination number of the transformation graph particularly in four different transformations - G^{+++}, G^{+-+}, G^{-++} and G^{--+}.

2. Preliminary results

The following results give the secure domination number of certain families of graphs that are required for further discussion.

Theorem 2.1. [4] *For any graph G, $\gamma_s(G) = 1$ if and only if $G = K_n$.*

Theorem 2.2. [4]

(1) *For a path P_n, $\gamma_s(P_n) = \lceil \frac{3n}{7} \rceil$.*
(2) *For a cycle C_n, $\gamma_s(C_n) = \lceil \frac{3n}{7} \rceil$.*
(3) *For a star $K_{1,n}$, $\gamma_s(K_{1,n}) = n$.*

Cockayne et al. [7] characterized secure dominating set in terms of the concept of external private neighborhood of a vertex. The following results give the necessary and sufficient conditions for connected graphs to have secure domination number equal to 2.

Theorem 2.3. [7] *If S is a non-empty set and $S \subseteq V(G)$ then the following statements are equivalent:*
(i) S is a secure dominating set.
(ii) For each $u \in V(G) \setminus S$, there exists $v \in S \cap N(u)$ such that $epn(v, S) \subseteq N[u]$.
(iii) For each $u \in V(G) \setminus S$, there exists $v \in S \cap N(u)$ such that the induced subgraph of $\{u, v\} \cup epn(v, S)$ is complete.

Theorem 2.4. [7] *If G is a graph with $n \geq 3$ then $\gamma_s(G) = 2$ if and only if G is non-complete and there exist distinct vertices u and v that dominate G and satisfy one of the following conditions:*
(i) $N(u) \setminus \{v\} = N(v) \setminus \{u\} = V(G) \setminus \{u, v\}$.
(ii) The induced subgraph of $(N(u) \setminus N[v])$ and $(N(v) \setminus N[u])$ are complete and for each $x \in N(u) \cap N(v)$ either the induced subgraph of $(N(u) \setminus N[v]) \cup \{x\}$ or the induced subgraph of $(N(v) \setminus N[u]) \cup \{x\}$ is complete.
(iii) $N(u) \setminus \{v\} = V(G) \setminus \{u, v\}$, $(N(u) \setminus N[v])$ is a non-empty set and the induced subgraph of $N(u) \setminus N(v)$ is complete.

Theorem 2.5. [6] *For any graph G, $\gamma(G^{+-+}) = 1$ if and only if $G \cong K_{1,r}$, $r \geq 1$.*

Recall that for any graph G, $\gamma(G) \leq \gamma_s(G)$. [4] .

Theorem 2.6. [1] *If G is graph with n vertices and m edges then $\gamma(G^{-++}) \leq 1 + \gamma(L(G))$.*

3. Bounds for secure domination number in Transformation graphs G^{xy+}

The first result gives the necessary and sufficient condition for a graph G with secure domination number of the transformation graph equal to 1.

Theorem 3.1. *For any connected graph G, $\gamma_s(G^{+y+}) = 1$ if and only if G $= K_2$.*

Proof. Assume that $\gamma_s(G^{+y+}) = 1$. Let S be the minimum secure dominating set of G^{+y+}. If S = $\{e\}$ where $e \in$ E(G), then e is incident with exactly two vertices of G. Now from Theorem 2, it is understood that G^{+y+} must be complete and so $G \cong K_2$. If S = $\{v\}$ where $v \in$ V(G), then v must be adjacent to all vertices and incident with all edges of G so to form a complete graph in G^{+y+} which implies $G \cong K_2$.
Conversely, let $G = K_2$. Then $K_2^{+y+} \cong K_3$ and γ_s $(K_3) = 1$. □

Note that for any graph G, $1 \leq \gamma_s(G^{xy+}) \leq$ n. Also for a connected graph G, γ_s $(G^{-y+}) \geq 2$. The bound in case of G^{-y+} is sharp when G = K_2. Table 1 gives the graphs corresponding to each transformation graph G^{xy+} where $\gamma_s(G^{xy+}) = 1$.

	γ_s $(G^{xy+}) = 1$			
xy	+ +	+ -	- +	- -
G	K_2	K_2	-	-

TABLE 1. Graphs where $\gamma_s(G^{xy+})$ is 1

The next result gives the value of the parameter G^{xy+} over all but some positive integers.

Proposition 3.2. *If n and p are two integers such that $n \geq 5$ and $3 \leq p <$ n - 1, then there exists a connected graph G where $\gamma_s(G^{+++}) = p$.*

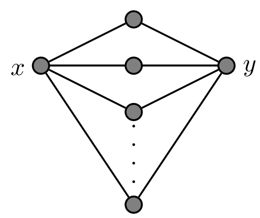

FIGURE 2. Uniform theta graph $\theta(n - (p - 1), 1)$

Proof. Consider the theta graph $\theta(n - (p - 1), 1)$ with $n \geq 5$ as in Figure 2, where $u_1, u_2,\ldots$, and $u_{n-(p-1)}$ are the vertices incident to both x and y. Let e be an edge incident to x and u_i. In this graph, add $(p - 3)$ isolated vertices, say, $v_1, v_2,\ldots$, and v_{p-3} where $3 \leq p < n - 1$ to y and then add edges yv_i for each $1 \leq i \leq p - 3$ to form the graph G. Then the set $S = \{x, y, e, v_1, v_2,\ldots,v_{p-3}\}$ will form a secure dominating set of G^{+++} with $|S|$ $= p$. □

Proposition 3.3. *If n and p are integers such that $n \geq 5$ and $4 \leq p \leq n$ - 1, then there exists a connected graph G where $\gamma_s(G^{+-+}) = p$.*

Proof. Consider the theta graph $\theta(n - (p - 2), 1)$ with $n \geq 5$ as in Figure 2 where $u_1, u_2,\ldots$, and $u_{n-(p-2)}$ are the vertices incident to both x and y. Let e be an edge incident to x and u_i and f be an edge incident to y and u_i. In this graph add $(p-4)$ isolated vertices say $v_1, v_2,\ldots$, and v_{p-4} where $4 \leq$ p $\leq n-1$ to u_1 and then add edges u_1v_i for each $1 \leq i \leq p-3$ to form the graph G. Then the set $S = \{x, y, e, f, v_1, v_2, \ldots, v_{p-4}\}$ will form a secure dominating set of G^{+-+} with $|S| = p$. □

Proposition 3.4. *If any two integers p and n are such that $1 \leq p < n$ and $p \neq 2$, then there exists a graph G where $\gamma_s(G^{--+}) = p$.*

Proof. Consider $K_{1,p-1}$ whose pendant vertices are $u_1, u_2, \ldots$, and u_{p-1} and $(n-p)$ isolated vertices say $v_1, v_2, \ldots$, and v_{n-p} then v_1, $u_1, u_2, \ldots$, and u_{p-1} forms a secure dominating set of G^{--+}. □

The construction mentioned in Propositions 3.2, 3.3 and 3.4 are not unique and these families are not exclusive. The next few theorems gives the necessary and sufficient condition for a graph G with secure domination number of the transformation graph equal to 2.

Lemma 3.5. *If G consists of at least 3 pendant vertices, then $\gamma_s(G^{+++}) > 2$.*

Proof. Let p be a pendant vertex with a root vertex v and let e be the edge incident to p and v in G. This implies either p, v or e must be in secure dominating set which securely dominates only these three vertices in G^{+++}. Hence, if G consists of minimum 3 pendant vertices then clearly γ_s set contains at least 3 vertices of G^{+++} and hence, $\gamma_s(G^{+++}) > 2$. □

Theorem 3.6. *For any graph G with $n \geq 3$, $\gamma_s(G^{+++}) = 2$ if and only if G is either P_3 or C_3 or a Paw.*

Proof. We know that γ(G) $\leq \gamma_s$(G) [4]. Hence, $\gamma_s(G^{+++}) = 2$ if either $\gamma(G^{+++}) = 1$ or $\gamma(G^{+++}) = 2$. We consider the following cases.

Case (1): $\gamma(G^{+++}) = 1$.

In this case there exists a vertex v such that $deg(v) = n + m - 1$ in G^{+++}. Therefore, $v \notin E(G)$ since v is adjacent to only 2 vertices and $n \geq 3$. Hence, $v \in V(G)$ and $G = K_{1,n}$. But from Lemma 3.5, $G = K_{1,2}$. This implies if $\gamma_s(G^{+++}) = 2$ then, $G = K_{1,2}$ (P_3).

Case (2): $\gamma(G^{+++}) = 2$.

Let $S = \{u, v\}$ be a dominating in G^{+++}.

Subcase (1): u,$v \in E(G)$.

Let $v_1, v_2, \ldots$, and v_n be the vertices and let $e_1, e_2, \ldots, e_{m-1}, u$, and v be the

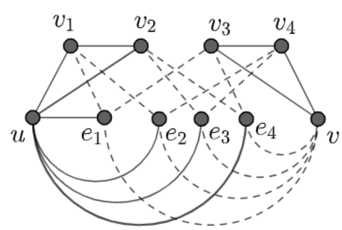

FIGURE 3. Transformation graph G^{+++} where $n = 4$ vertices in G

edges of G where u is an edge which is adjacent to the maximum number of edges in G^{+++} . Clearly, $n \leq 4$ since G is a connected graph and an edge is

incident to exactly two vertices in G^{+++}.
If u is not dominated by a single vertex v in G^{+++} then, $L(G)$ is a complete graph and the graph G^{+++} has exactly 3 vertices. This implies $G = C_3$ and this S also forms a secure dominating set in G^{+++}. Hence, $\gamma\ (G^{+++}) = 2$ if $G = C_3$.
If u is not dominated by 2 vertices and an edge then, $epn(u, S)$ is a set with v_1, v_2 and at least 1 edge of G in G^{+++} which is not a contained in $N[v_1]$. See Figure 3. Hence, S cannot form a secure dominating set of G^{+++}.

Subcase (2): u, $v \in V(G)$.
Let $v_1, v_2, \ldots, v_{n-2}, u$, and v be the vertices and let $e_1, e_2, \ldots,$ and e_m be the edges of G where u is the vertex with $deg(u) = \Delta(G)$ in G^{+++}.
If u is not dominated by a single edge in G^{+++} then, this vertex is incident to m - 1 edges in G as in Figure 4. Notice that if $\gamma_s(G^{+++}) \leq 2$ then, G

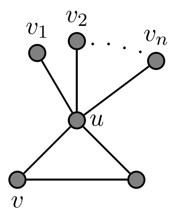

FIGURE 4. Graph G with n pendant vertices.

consists of pendant vertices less than 3, from Lemma 3.5. This implies $n = 2$ say v_1 and v_2. But for v_1, $epn(u, S)$ is a set with v_2 and at least 1 edge of G in G^{+++} which is not contained in $N[v_1]$. Therefore $n = 1$ and hence, if $\gamma(G^{+++}) = 2$ then G is a Paw.
If u is not dominated by a vertex v and an edge e_m in G^{+++} then, for v_1, $epn(S, u) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{m-1}\} \nsubseteq N[v_1]$. See Figure 5. Therefore, S cannot form a secure dominating set in G^{+++} and this is the same when u is not dominated by vertices and edges greater than or equal to 1.

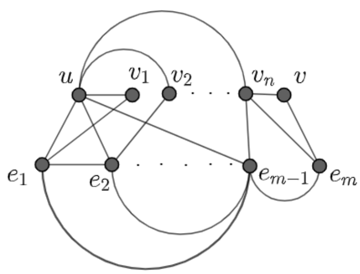

FIGURE 5. Transformation graph G^{+++} with $\gamma\ (G^{+++}) = 2$

Subcase (3): $u \in V(G)$, $v \in E(G)$.
Let u be the vertex with $deg(u) = \Delta$(G). This proof is similar to Subcase (2) since u is same in both cases.
Conversely, if $G = C_3$ then C_3^{+++} satisfies case (1) of Theorem 2.4 and

hence, $\gamma_s(C_3^{+++}) = 2$. If $G = P_3$ or a Paw then, both the graphs satisfies case (3) of Theorem 2.4. Therefore, the secure domination number of P_3 and a Paw is 2. □

Theorem 3.7. *For a connected graph G with $n \geq 3$, $\gamma_s(G^{+-+}) = 2$ if and only if $G = P_3$.*

Proof. We know that γ(G) $\leq \gamma_s$(G) [4]. Hence, $\gamma_s(G^{+-+}) = 2$ if either $\gamma(G^{+-+}) = 1$ or $\gamma(G^{+-+}) = 2$. We consider the following cases.
Case (1): $\gamma\ (G^{+-+}) = 1$.
It is clear that $\gamma\ (G^{+-+}) = 1$ if and only if $G = K_{1,n}$, from Theorem 2.5. Let S be the secure dominating set and let v be the vertex in the γ set whose degree is n - 1. If $v \in S$ then, since S is a secure dominating set of G^{+-+}, $|S| \geq n$. If $v \notin S$, then, for each $v_i \in$ V$(G^{+-+}) \setminus S$ there exists $v \in S$ such that *epn* $(v$,S$) \nsubseteq N_G(v_i)$. Therefore, $|S| \geq n$. Since S is a secure dominating set of G^{+-+}, $\gamma_s(K_{1,n}^{+-+}) = n$.

Case (2): $\gamma\ (G^{+-+}) = 2$.
Let $S = \{u, v\}$ be a dominating in G^{+-+}. Clearly, V$(G^{+-+}) > 5$.

Subcase (1): u, $v \in E(G)$.
In this case $n = 4$ since $V(G^{+-+}) > 5$. If u, $v \in E(G)$ then S is not a dominating set of G^{+-+} since y = - in G^{xy+}.

Subcase (2): u, $v \in V(G)$ or $u \in V(G)$ and $v \in E(G)$.
In this case there exists at least one edge not in S which does not satisfy case (2) of Theorem 2.3. Hence, S is not a secure dominating set of G^{+-+}.

Conversely, if $G = P_3$ then P_3^{+-+} satisfies case (2) of Theorem 2.4. Hence, $\gamma_s\ (G^{+-+}) = 2$. □

Theorem 3.8. *For a connected graph G, $\gamma_s(G^{--+}) = 2$ if and only if $G = K_2$.*

Proof. We know that γ(G) $\leq \gamma_s$(G) [4]. Hence, $\gamma_s(G^{--+}) = 2$ if either $\gamma(G^{--+}) = 1$ or $\gamma(G^{--+}) = 2$. We consider the following cases.
Case (1): $\gamma(G^{--+}) = 1$.
In this case there exists a vertex v such that $deg(v) = n + m - 1$ in G^{--+}. Therefore, $v \notin V(G)$ since G is connected and there exists at least one vertex $u \in V(G)$ which is not adjacent to v in G^{--+}. This implies $v \in E(G)$ and this edge is incident to only two vertices say v_1 and v_2 in G^{--+}. Hence, the only case where $\gamma\ (G^{--+}) = 1$ is when a graph consists of a single edge which is K_2. Now, since there are two external private neighbours for v that is v_1 and v_2, v cannot be the γ_s set of K_2^{--+} and hence, $\gamma_s\ (K_2^{--+}) = 2$. Therefore, if $\gamma_s\ (G^{--+}) = 2$ then, $G = K_2$.

Case (2): $\gamma(G^{--+}) = 2$.
Let $S = \{u, v\}$ be a dominating in G^{--+}.

Subcase (1): u, $v \in E(G)$.
In this case G is connected only if $n = 3$ and also there are exactly two edges u, v in G^{--+} else S will not be a dominating set of G^{--+}. This implies that if u, $v \in E(G)$ and $\gamma\ (G^{--+}) = 2$ only if $G = P_3$. $P_3^{--+} \cong C_5$ whose secure domination number is equal to 3 from Theorem 2.2.

Subcase (2): u, $v \in V(G)$.
Let $v_1, v_2, \ldots, v_{n-2}, u$, and v be the vertices and let $e_1, e_2, \ldots,$ and e_m be the edges of G. If u is incident to all edges in G and v is an end vertex of

any edge in G then, this forms a dominating set in G^{--+}. See Figure 6. But for $e_1 \in V(G) \setminus S$ there exists $u \in S$ such that $< \{u, v\} \cup epn(v, S) >$

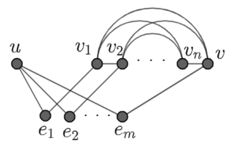

FIGURE 6. Transformation graph G^{--+} with $\gamma(G^{--+}) = 2$

is not complete. Hence, S is not a secure dominating set of G^{--+}.

Subcase (3): $u \in V(G)$ and $v \in$ E(G).

Since $v \in$ E(G), v is incident to exactly two vertices say v_1 and v_2 in G^{--+}. Hence, $V(G) = \{u, v_1, v_2\}$ otherwise G is not connected. Now u can be adjacent to either v_1 or v_2 or both. If v is adjacent to both v_1 and v_2 then, $G \cong C_3$, $C_3^{--+} = C_6$ whose secure domination number is 3 from Theorem 2.2. If v is adjacent to either v_1 or v_2 then, $G \cong P_3$ and $\gamma_s(P_3) \neq 2$ from Subcase (1). Hence, S is not a secure dominating set of G^{--+}.

Conversely, if $G = K_2$ then, $K_2^{--+} \cong P_3$ and $\gamma_s(P_3) = 2$ from Theorem 2.2.

□

Theorem 3.9. *If G is a connected graph, then $\gamma_s(G^{-++}) = 2$ if and only if $G = K_{1,n}$.*

Proof. We know that γ(G) $\leq \gamma_s$(G) [4]. Hence, $\gamma_s(G^{-++}) = 2$ if either $\gamma(G^{-++}) = 1$ or $\gamma(G^{-++}) = 2$. We consider the following cases.

Case (1): $\gamma(G^{-++}) = 1$.

In this case there exists a vertex v such that $deg(v) = n + m - 1$ in G^{-++}. Therefore, $v \notin V(G)$ since G is connected and there exists at least one vertex $u \in V(G)$ which is not adjacent to v in G^{-++}. This implies $v \in E(G)$ and this edge is incident to only two vertices say, v_1 and v_2 in G^{-++}. Hence, the only case where γ $(G^{-++}) = 1$ is when a graph consists of a single edge which is K_2. Now since there are two external private neighbours for v that is v_1 and v_2, v cannot be γ_s set of K_2^{-++}. Hence, γ_s $(K_2^{-++}) = 2$. Therefore, if γ_s $(G^{-++}) = 2$ then, G $= K_2$.

Case (2): $\gamma(G^{-++}) = 2$.

Let $S = \{u, v\}$ be a dominating in G^{-++}.

Subcase (1): u, $v \in E(G)$.

In this case since G is connected $n = 3$ or $n = 4$ but since $\overline{G}$ is a subgraph of G^{-++}. Therefore, S is not a secure dominating set of G^{-++}.

Subcase (2): u, $v \in V(G)$.

Let u be the vertex non- adjacent to vertices $u_1, u_2, \ldots,$ and u_p and adjacent to vertices $v, v_1, v_2, \ldots,$ and v_q and let e_1, e_2, $\ldots$, and e_p be the edges incident to u in G. Then clearly, u dominates $u_1, u_2, \ldots, u_p$, $e_1, e_2, \ldots,$ and e_q as in Figure 7 .

Now v can dominate the rest of the vertices of G^{--+} only if they are non-adjacent in G. Hence, $v, v_1, v_2, ..., v_q$ form a complete graph in G^{-++}. But for each u_i not in S, $epn(u, S) = \{u_1, u_2, \ldots, u_p, e_1, e_2, \ldots, e_p\} \nsubseteq$ N$[u_i]$.

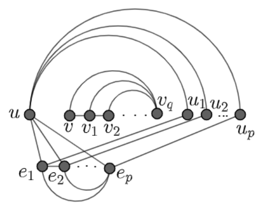

FIGURE 7. Transformation graph G^{-++} with $\gamma(G^{-++}) = 2$

Hence, S is a secure dominating set of G^{-++} if the vertices $u_1, u_2, ..., u_p$ are not in G. This implies $G = K_{1,n}$, $n > 1$. Hence, if $\gamma_s(G^{-++}) = 2$ then $G = K_{1,n}$, $n > 1$.

Subcase (3): $u \in V(G)$ and $v \in E(G)$.

In this case $u = e_1$ and $v = u$ in S since $u \in E(G)$. See Figure 7 . Then, from Subcase (2), S is a secure dominating set if $u_1, u_2, \ldots,$ and u_p are not in G and hence, $G = K_{1,n}$, $n > 1$. Suppose $v, v_1, v_2, ..., v_q$ do not form a complete graph in G^{-++} then, S is a dominating set if v and v_1 is adjacent with an edge say, e. Then, u and $v = e$ forms a dominating set of G^{-++}. But $epn(e, S) = \{v, v_1\} \nsubseteq \mathrm{N}[v]$. Hence, S is not a secure dominating set of G. In the case of a bistar, any pendant vertex v and the edge e adjacent to the root vertices are in S. But this is not a secure dominating set of G^{-++} since $epn(e, S)$ consists of v and all pendant edges which is not contained in $\mathrm{N}[e_1]$, where e_1 is a pendant edge in G. Hence, S is not a secure dominating set of G^{-++} if $u \in V(G)$ and $v \in E(G)$.

From case (1) and (2) we can conclude that if $\gamma_s(G^{-++}) = 2$ then, $G = K_{1,n}$.

Conversely, if $G = K_{1,n}$ then G satisfies the condition of Theorem 2.4. Hence, $\gamma_s(K_{1,n}^{-++}) = 2$. □

Table 2 gives the graphs corresponding to each transformation graph G^{xy+} where $\gamma_s(G^{xy+}) = 2$.

	$\gamma_s(G^{xy+}) = 2$			
xy	+ +	+ -	- +	- -
G	P_3, C_3 or Paw	P_3	$K_{1,n}$	K_2

TABLE 2. Graphs where $\gamma_s(G^{xy+})$ is 2

It can be noted that the given graph G is a subgraph of G^{+yz}, $L(G)$ is a subgraph of G^{x+z} and the subdivision graph is a subgraph of G^{x+}. The following results give bounds in terms of these subgraphs.

Theorem 3.10. *Let G be a connected graph, then $\gamma_s(G^{-++}) \leq \gamma_s(L(G))$ + 2.*

Proof. Clearly, $\gamma(G^{-++}) \leq \gamma(L(G)) + 1$ from Theorem 2.6. Let $S \subseteq V(G^{-++})$ and $S = S_1 \cup S_2$ where $n(S_1) = \gamma_s(L(G)) + 1$. But for each $a_i \in V(G^{-++}) \setminus S_1$ if there exists $v_i \in S_1 \cap N(a_i)$ such that $epn(v_i, S_1)$ $= \{a_i, b_i\} \nsubseteq N_G[a_i]$ then, let S_2 be the set of all b_i's. This set of vertices of S_2 forms a complete graph. Hence, the vertices of G^{-++} are securely dominated by the set S and $\gamma_s(G^{-++}) \leq \gamma_s(L(G)) + 1 + 1$.

The inequality is sharp in the case of P_8. □

Note that for any connected graph G, $\gamma_s(\text{G}) \leq \gamma_s(G^{+y+})$.

Lemma 3.11. *If G is a connected graph with at least two pendant vertices, then $\gamma_s(G) \geq \gamma_s(\overline{G})$.*

Proof. Let D and D' be the minimum secure dominating set of G and $\overline{G}$, respectively. If G is not a tree then D' contains any two pendant vertices and one root vertex of G since G contains at least 2 pendant vertices. But D contains all pendant vertices and at least one vertex of G which implies $\gamma_s(G) \geq \gamma_s(\overline{G})$. If G is a tree then $\gamma_s(\overline{G}) = 2$ which implies $\gamma_s(\text{G}) \geq \gamma_s(\overline{G})$ for a graph G with at least two pendant vertices in G. □

Theorem 3.12. *If G consists of at least 4 pendant vertices, then $\gamma_s(G) \geq \gamma_s(G^{-++})$.*

Proof. Assume the contrary that if G consists of minimum 4 pendant vertices then $\gamma_s(G) < \gamma_s(G^{-++})$. Let S and S' be the minimum secure dominating set of G and G^{-++} respectively then $|S'| > |S|$.

If edges of G are not in S'. Then clearly, S' is a secure dominating set of $\overline{G}$ and from Lemma 3.11, $|S'| > |S|$ is a contradiction.

If edges of G are in S' then let $\{e_1, e_2, \ldots, e_k\}$ where $e_i = u_iv_i$ be the edges in S' and let $\{p_1, p_2, \ldots, p_l\}$ where $l \geq 4$, be the pendant vertices in G^{-++}.

Case (1): *All the pendant vertices are incident to a single vertex u in G.* It is easy to note that the set $P = \{p_1, p_2, \ldots, p_l\}$ and $E = \{e_{p1}, e_{p2}, \ldots, e_{pl}, u\}$ forms complete graphs in G^{-++} and hence, one vertex from each set must be in S'. Now $[S' - \{e_1, e_2, \ldots, e_k\}] \cup \{u_1, u_2, \ldots, u_k\}]$ form a secure dominating set in G^{-++} and $\overline{G}$ since each edge is adjacent to two vertices of G. From Lemma 3.11 this is a contradiction.

Case (2): *All the pendant vertices are not incident to a single vertex in G.*

In this case P will be a complete graph but E will not be a complete graph in G^{-++}. This implies that either S' contains all or few edges incident to the pendant vertices or neither edges incident to the pendant vertices. In all cases $|S'| > |S|$ is a contradiction from Lemma 3.11.

Further for the graph G in Figure 8, the bound is sharp since $\gamma_s(G) =$

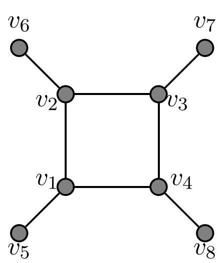

FIGURE 8. Graph $C_4 \circ K_1$ where $\gamma_s(\text{G}) = \gamma_s(G^{-++})$

$\gamma_s(G^{-++}) = 4$. □

Lemma 3.13. *If T is a tree with $\Delta(T)$ = n - 2, then $\gamma_s(T^{--+}) = 3$.*

Proof. Let u be the vertex with maximum degree $\Delta(T)$ and v be the vertex such that $v \notin N[u]$. Let$f$ be the edge incident to v and any vertex adjacent to u say, u_i. Then, $S = \{u, u_i\}$ is a dominating set in T^{--+}. But since an edge is adjacent to only two vertices in T^{--+}, S is not a secure dominating set of T^{--+}. Therefore, $S = \{u, v, e\}$ will form a secure dominating set of T^{--+} and $|S| = 3$ which implies $\gamma_s(T^{--+}) = 3$. □

Theorem 3.14. *If T is a tree with $n \geq 8$, then $\gamma_s(T) \geq \gamma_s(T^{--+})$.*

Proof. Let S and S' be the minimum secure dominating set of T and T^{--+}, respectively and let u be the vertex with maximum degree $\Delta(T)$.

Case (1): $\Delta(T) = n - 1$.
In this case, $T = K_{1,n-1}$ and from Theorem 2.2, $\gamma_s(T) = n - 1$. If $u \in S'$ then, since S' is a secure dominating set of T^{--+}, $|T| \geq n - 1$. If $u \notin S'$, then, for each $v_i \in V(T^{--+}) \setminus S'$ there exists $u \in S'$ such that $epn(u,S')$ = $\{v_1, v_2, \ldots, v_{n-1}\} \nsubseteq N_G[v_i]$. Therefore, $|S'| \geq n$ - 1. Since S' is a secure dominating set of T^{--+}, $\gamma_s(T^{--+}) = n - 1$. Hence, if $\Delta(T) = n - 1$ then $\gamma_s(T) = \gamma_s(T^{--+})$.

Case (2): $\Delta(T) = n - 2$.
Clearly, $\gamma_s(T) = n$ - 2. It follows from Lemma 3.13 that $\gamma_s(T) > \gamma_s(T^{--+})$.

Case (3): $\Delta(T) < n$ - 2
Assume the contrary that if T is a tree with $n \geq 8$ then $\gamma_s(T) < \gamma_s(T^{--+})$. This implies $|S| < |S'|$. If edges of T are not in S' then S' is a secure dominating set of $\overline{T}$ and from Lemma 3.11, $|S| < |S'|$ is a contradiction. Suppose edges of T are in S' then, let $\{e_1, e_2, \ldots, e_k\}$ where $e_i = u_i v_i$ be the edges in S'. Then $[S' - \{e_1, e_2, \ldots, e_k\}] \cup \{u_1, u_2, \ldots, u_k\}]$ forms a secure dominating set in T^{--+} and $\overline{T}$. Therefore, from Lemma 3.11, $|S| <$ $|S'|$ is a contradiction. Hence, γ_s(T) $\geq \gamma_s(T^{--+})$. □

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